

3-groups are not determined by their integral cohomology rings

Ian Leary
 Departement Mathematik,
 ETH Zentrum,
 8092 Zürich.

Abstract.

There is exactly one compact 1-dimensional Lie group having 27 components and nilpotence class three. We give a presentation for the integral cohomology ring of (the classifying space of) this group. We show that the groups of order 3^4 can be distinguished by their first few integral cohomology groups, and exhibit a pair of groups of order 3^5 having isomorphic integral cohomology rings.

Introduction

The aim of this paper is the study of the integral cohomology rings of a family of 3-groups. For each $n \geq 4$ and $e = \pm 1$ a member $G(n, e)$ of this family is defined. The group $G(n, e)$ has order 3^n and may be presented as follows.

$$G(n, e) = \langle A, B, C \mid A^3 = B^{3^{n-2}} = C^3 = [B, C] = 1, [B, A] = C, [C, A] = B^{e3^{n-3}} \rangle$$

The groups $G(n, 1)$ and $G(n, -1)$ are not isomorphic to each other. The main result of this paper (Corollary 14) is that for $n \geq 5$ the integral cohomology rings of $G(n, 1)$ and $G(n, -1)$ are isomorphic. These seem to be the first examples of p -groups for any p having this property. An elegant argument due to Alperin and Atiyah ([5], page 86) establishes the existence of groups whose orders divide by more than one prime having isomorphic integral cohomology rings, and metacyclic examples are known [8]. For p -groups there seems to be no way to exhibit such groups without actually determining the relevant cohomology rings and showing that they are isomorphic, and this is what we do. (See however [10] for a non-computational method to exhibit p -groups having isomorphic integral cohomology groups.)

The groups $G(n, e)$ all occur as normal subgroups of a single compact 1-dimensional Lie group \tilde{G} such that the quotient group is connected, and our method involves first finding the cohomology of the Lie group \tilde{G} . For each prime $p \geq 5$ and $n \geq 4$ two isomorphism types of groups of order p^n having similar presentations to those given above for $G(n, e)$ may be defined. Using the methods of this paper we have obtained some partial results concerning the cohomology of these groups. These partial results have also been obtained by N. Yagita using different methods, so here we merely indicate how they could be proved.

The first section of the paper consists of a brief discussion of the method we employ, the second is an examination of the groups $G(n, e)$, \tilde{G} and various subgroups and quotients, and the third section contains the main results and proofs.

Method.

Let \mathbf{T} be the group of complex numbers of modulus one. If G is a finite group and Z a central cyclic subgroup of G then we define \tilde{G} to be the central product of T and G amalgamating Z with the isomorphic subgroup of T (via some fixed embedding). The group \tilde{G} is a compact Lie group with identity component isomorphic to \mathbf{T} and group of components G/Z . The group G is a normal subgroup of \tilde{G} with quotient $T/Z \cong \mathbf{T}$, and since T is central in \tilde{G} , BG is a principal \mathbf{T} -bundle over $B\tilde{G}$. Under the natural isomorphism $H^2(\tilde{G}) \cong \text{Hom}(\tilde{G}, \mathbf{T})$, the first Chern class of this bundle corresponds to a morphism with kernel G . This construction is explained in more detail in [9], and was suggested by P. H. Kropholler and J. Huebschmann [6, 7].

For any group K , let $\text{Ch}(K)$ be the subring of $H^*(K)$ generated by Chern classes of complex representations of K , and let $\overline{\text{Ch}}(K)$ be its ‘‘Mackey closure’’, that is the subring of $H^*(K)$ generated by $\text{Ch}(K)$ and the images under the transfer of $\text{Ch}(H)$ as H ranges over the finite index subgroups of K . One may ask under what circumstances either $\text{Ch}(K)$ or $\overline{\text{Ch}}(K)$ is the whole of the even degree cohomology of K . If G and \tilde{G} are as above, then the following lemma links these properties for G and \tilde{G} .

Lemma 1. *Let G be a finite group with central cyclic subgroup Z and construct \tilde{G} as above. Then $\text{Ch}(G) = H^{\text{even}}(G)$ (resp. $\overline{\text{Ch}}(G) = H^{\text{even}}(G)$) if and only if $\text{Ch}(\tilde{G}) = H^{\text{even}}(\tilde{G})$ (resp. $\overline{\text{Ch}}(\tilde{G}) = H^{\text{even}}(\tilde{G})$) and multiplication by the Chern class of BG as a bundle over $B\tilde{G}$ is injective on the odd degree cohomology of \tilde{G} .*

Proof. First we claim that any complex representation of G extends to one of \tilde{G} . This follows from the fact that Z must act via scalar multiplication in any irreducible G -representation, because an eigenspace for any central element of G is a G -summand. Thus the action may be extended from Z to T so that T also acts by scalar multiplication, and because the image of T is central in $\text{End}(V)$ the action extends to \tilde{G} .

It follows that the image of $\text{Ch}(\tilde{G})$ in $H^*(G)$ is exactly $\text{Ch}(G)$. Subgroups of G are in one-to-one correspondence with finite index subgroups of \tilde{G} . If \tilde{H} is a finite index subgroup of \tilde{G} , with $H = \tilde{H} \cap G$ the corresponding subgroup of G , then $\text{Res}_{\tilde{G}}^{\tilde{H}} \text{Cor}_{\tilde{H}}^{\tilde{G}} = \text{Cor}_H^G \text{Res}_H^{\tilde{H}}$ because $G\tilde{H} = \tilde{G}$, and it follows that the image of $\overline{\text{Ch}}(\tilde{G})$ is $\overline{\text{Ch}}(G)$. From now on the proofs of the statements concerning $\text{Ch}(G)$ and $\overline{\text{Ch}}(G)$ are identical, so we consider only the former.

Now consider the spectral sequence for BG as a \mathbf{T} -bundle over $B\tilde{G}$. In this spectral sequence the group $E_3^{n,1}$ is isomorphic to the cokernel of the map from $H^{n+1}(\tilde{G})$ to $H^{n+1}(G)$, and is equal to the kernel of multiplication by the Chern class of the bundle as a map from $H^n(\tilde{G})$ to $H^{n+2}(\tilde{G})$, so that $H^{\text{even}}(\tilde{G})$ maps onto $H^{\text{even}}(G)$ if and only if multiplication by this element is injective on $H^{\text{odd}}(\tilde{G})$. Now if $\text{Ch}(\tilde{G}) = H^{\text{even}}(\tilde{G})$, then $\text{Ch}(\tilde{G})$ and hence also $\text{Ch}(G)$ map onto $E_3^{\text{even},0}$, and so in this case $\text{Ch}(G) = H^{\text{even}}(G)$ if and only if $E_3^{\text{odd},1}$ is trivial. It remains to consider the case when $\text{Ch}(\tilde{G})$ is not the whole of $H^{\text{even}}(\tilde{G})$. In this case pick x of minimal degree in $H^{\text{even}}(\tilde{G}) \setminus \text{Ch}(\tilde{G})$. Since $H^2(\tilde{G}) \cong \text{Hom}(\tilde{G}, \mathbf{T})$ is contained in $\text{Ch}(\tilde{G})$, x has degree at least four. If it were the case that $\text{Ch}(G) = H^{\text{even}}(G)$, then x would have to be congruent to an element of $\text{Ch}(\tilde{G})$ modulo the image of the differential, that is $x = z + cy$, where c is the Chern class of BG as a bundle over $B\tilde{G}$, and z is an element of $\text{Ch}(\tilde{G})$. But now y and c have lower degree than x , so are also in $\text{Ch}(\tilde{G})$, and so we obtain a contradiction. •

The groups $G(n, e)$ and \tilde{G} .

The group $G = G(n, e)$ as presented in the introduction is generated by two elements, because the element C is already in the subgroup generated by A and B . It follows that the quotient of G by the intersection of its maximal subgroups is elementary abelian of rank two, and hence that G has exactly four maximal subgroups. The intersection of these is the subgroup generated by B^3 and C , and is isomorphic to $C_{3^{n-3}} \oplus C_3$. We shall call this subgroup N , or $N(n, e)$. Any element not in N is contained in a unique maximal subgroup. The maximal subgroup containing B is isomorphic to $C_{3^{n-2}} \oplus C_3$, and will be referred to as M or $M(n, e)$. The maximal subgroup containing A is a non-abelian non-metacyclic group expressible as a central extension with kernel $C_{3^{n-3}}$ and quotient $C_3 \oplus C_3$ (there is a unique isomorphism type of group having these properties). We shall call this subgroup P . Note that the intersection of P and M is N . The other two maximal subgroups, those containing AB and AB^2 , are non-abelian metacyclic groups with a cyclic subgroup of index three, except in the case of $G(4, -1)$ when they are isomorphic to $P(4, -1)$, which is the non-abelian group of order 27 and exponent 3.

The centre of $G(n, e)$ is cyclic of order 3^{n-3} generated by B^3 . The quotient of $G(n, e)$ by its centre shall be called $E(n, e)$, and is the (unique) non-abelian group of order 27 and exponent 3. Since any order 3 normal subgroup of a 3-group must be central, it follows that $G(n, e)$ has a unique such subgroup, the subgroup generated by B^{n-3} . The quotient groups $G(n, 1)/\langle B^{n-3} \rangle$ and $G(n, -1)/\langle B^{n-3} \rangle$ are isomorphic, via a map sending the elements A, B and C in one group to the elements of the same name in the other, and it follows that the lattices of normal subgroups of $G(n, 1)$ and $G(n, -1)$ are isomorphic. If n is at least 5 then a far stronger statement is true. In this case, if we write elements of $G(n, e)$ as words of the form $A^i B^j C^k$ then a collection of elements forms a normal subgroup of $G(n, 1)$ if and only if the ‘same’ collection of elements forms a normal subgroup of $G(n, -1)$, and these two subgroups are isomorphic provided that they are proper. The author can think of no proof of this fact apart from direct calculation in the maximal subgroups.

It is known that the isomorphism type of a p -group is determined by that of its integral group ring [14, 18], and R. Sandling has shown the author a proof that the mod-3 group rings of the groups $G(n, 1)$ and $G(n, -1)$ are not isomorphic [16]. The group $G(n, e)$ has only 1- and 3-dimensional irreducible representations because it has an abelian subgroup (the subgroup M) of index three. The sizes of conjugacy classes in $G(n, 1)$ and $G(n, -1)$ are identical, but the character tables of $G(n, 1)$ and $G(n, -1)$ are different. The character table for $G(n, e)$ contains the entry $\eta(2 + \eta^{e3^{n-3}})$ for each primitive 3^{n-2} th root of unity η , while the character table for $G(n, -e)$ does not. This gives a proof that $G(n, 1)$ and $G(n, -1)$ are not isomorphic. (It is also reasonably straightforward to prove this fact directly.) The following proposition describes the Λ -ring structure of the representation ring of $G(n, e)$.

Proposition 2. *The representation ring of $G(n, e)$ is generated by θ, ψ of dimension one and $\chi, \bar{\chi}, \xi, \bar{\xi}$ of dimension three (where the bar indicates the dual representation) subject to the following relations, together with commutativity and the relations implied by $\overline{XY} = \bar{X}\bar{Y}$.*

$$\begin{aligned} \theta^3 &= 1, \quad \psi^{3^{n-3}} = 1, \quad \theta\chi = \chi, \quad \psi^{3^{n-4}}\chi = \chi, \\ \chi^2 &= 3\bar{\chi}, \quad \chi\bar{\chi} = (1 + \theta + \theta^2)(1 + \psi^{3^{n-4}} + \psi^{-3^{n-4}}), \\ \xi\chi &= \xi\bar{\chi} = \xi(1 + \psi^{3^{n-4}} + \psi^{-3^{n-4}}), \\ \theta\xi &= \xi, \quad \xi\bar{\xi} = \chi + \bar{\chi} + 1 + \theta + \theta^2, \\ \xi^2 &= \bar{\xi}\psi(1 + 2\psi^{e3^{n-4}}). \end{aligned}$$

The Λ -ring structure is given by the following equations.

$$\begin{aligned} \Lambda^2(\chi) &= \bar{\chi}\psi^3, \quad \Lambda^3(\chi) = \psi^3, \\ \Lambda^2(\xi) &= \bar{\xi}\psi^{(1+e3^{n-4})}, \quad \Lambda^3(\xi) = \psi^{(1+e3^{n-4})}. \end{aligned}$$

Proof. Direct calculation. In the above statement, θ, ψ and χ come from representations of the quotient of $G(n, e)$ by its central subgroup of order three, and ξ is a faithful representation of $G(n, e)$ obtained by inducing up a representation of $M(n, e)$ that is faithful on $\langle B \rangle$. •

I do not know if the representation rings of $G(n, 1)$ and $G(n, -1)$ are isomorphic. I conjecture that even after quotienting out by the ideal generated by 9 the rings are not isomorphic. This would imply by Atiyah's theorem [1] that the K -theory ring can distinguish the groups $G(n, 1)$ and $G(n, -1)$. For n at least 5, the representation rings modulo the ideal generated by 3 are isomorphic, via the map that sends ξ to $-\xi\psi^{2 \cdot 3^{n-5}}$, $\bar{\xi}$ to $-\bar{\xi}\psi^{7 \cdot 3^{n-5}}$, and fixes the other generators. This isomorphism commutes with taking duals but not with the Λ -ring structure.

One might hope to distinguish the representation rings of $G(n, 1)$ and $G(n, -1)$ by using their Λ -ring structure. For example, Grothendieck has defined a filtration (the γ -filtration) on any augmented Λ -ring, and one may consider the representation rings of the groups modulo the layers of this filtration. A theorem of Atiyah [1] and the fact that Chern classes generate the even degree cohomology of $G(n, e)$ (see Corollary 9) imply that this is equivalent to studying the image of $K^0(BG(n, e))$ in K^0 of its skeleta. The layers of the γ -filtration for $G(n, e)$ may be computed directly from Proposition 3, but the task is simplified by also using information concerning the low degree cohomology of $G(n, e)$ as provided by Theorem 13. The author has been able to show that the representation rings of $G(5, 1)$ and $G(5, -1)$ are isomorphic after quotienting out the third layer of the γ -filtration, or equivalently that the images of $K^0(BG(5, 1))$ and $K^0(BG(5, -1))$ in K^0 of their respective 5-skeleta are isomorphic.

There is one other finite group closely related to the groups $G(n, e)$. We may view $G(n, e)$ as an extension with kernel $M(n, e)$ and quotient cyclic of order three. In each case $G(n, e)$ is the corresponding split extension. It is easily verified that the second cohomology group of C_3 with coefficients in the module $M(n, e)$ is trivial, unless $n = 4$ and $e = -1$, in which case it has order three. This gives rise to another group of order 3^4 , which we shall call $G'(4)$. This group may be presented as follows.

$$G'(4) = \langle A, B, C | B^9 = C^3 = [B, C] = 1, [B, A] = C, [C, A] = B^{-3} = A^3 \rangle$$

The character table (and hence also the representation ring) of this group is identical to that of $G(4, -1)$, although the Λ -ring structure is slightly different.

Each of the groups $G(n, e)$ and $G'(4)$ is nilpotent of class three, and has cyclic centre of index 27 with quotient group isomorphic to E , the non-abelian group of order 27 and exponent 3. Now let G be any of the groups $G(n, e)$ or $G'(4)$, and apply the construction of section 1 to form a Lie group \tilde{G} as the central product of G and \mathbf{T} amalgamating the centre of G with the isomorphic subgroup of \mathbf{T} . In each case the resulting group is nilpotent of class three, has identity component isomorphic to \mathbf{T} and group of components E . There is however, only one isomorphism type of group having these properties, because the action of $\text{Aut}(E)$ on $H^2(E; \mathbf{T}) \cong H^3(E; \mathbf{Z})$ is transitive on non-zero elements, and $\mathbf{T} \times E$ is nilpotent of class two. Thus the construction of section 1 allows us to embed all of the groups $G(n, e)$ and $G'(4)$ into a single Lie group \tilde{G} as normal subgroups with connected quotient. This Lie group may be presented as follows, where \mathbf{T} is considered to be a subgroup of the complex numbers of modulus one, and ω is $\exp(2\pi i/3)$.

$$\tilde{G} = \langle X, Y, Z, \mathbf{T} | X^3 = Y^3 = Z^3 = 1, \mathbf{T} \text{ central}, [Y, Z] = 1, [Y, X] = Z, [Z, X] = \omega \rangle$$

This presentation enables one to express any element of \tilde{G} in the form $X^i Y^j Z^k t$ for some $t \in \mathbf{T}$. It is easy to check that if $\eta = \exp(2\pi i/3^{n-2})$, then the subgroup with generators $X, Y\eta^e$ and Z is isomorphic to $G(n, e)$ with generators A, B and C , and we fix this embedding from now on. The group \tilde{G} has four subgroups of index three, one of which is abelian, and three of which are the unique non-abelian Lie group consisting of nine circles. We shall refer to the subgroup generated by \mathbf{T}, Y and Z as \tilde{M} , this being the abelian subgroup of index three, and the subgroup generated by \mathbf{T}, X and Z shall be called \tilde{P} . Regarding the embedding of $G(n, e)$ in \tilde{G} as an inclusion the following equalities hold.

$$P(n, e) = \tilde{P} \cap G(n, e), \quad M(n, e) = \tilde{M} \cap G(n, e)$$

In the next section we shall require a description of some elements of $\text{Hom}(\tilde{G}, \mathbf{T})$ having kernel isomorphic to $G(n, e)$ or $G'(4)$. It is also possible to classify the isomorphism types of 3-groups that can occur as kernels of maps from \tilde{G} to \mathbf{T} , and so we combine these statements in the following proposition. Before stating the proposition, it is convenient to make the following definition, which we shall use frequently in the sequel.

Definition. For any ring R and any R -module M , we say that a subset S is a basis for M if zero is not an element of S and M is isomorphic to the direct sum of the submodules generated by the elements of S . Note that any finitely generated module for a principal ideal domain has such a basis.

Proposition 3. Define α (resp. β , δ_1) in $\text{Hom}(\tilde{G}, \mathbf{T})$ by insisting that it maps $X^i Y^j Z^k t$ to ω^i (resp. ω^j , t^3). Then α and β have order three, δ_1 has infinite order, and these elements form a basis for the group $\text{Hom}(\tilde{G}, \mathbf{T})$. The elements of $\text{Hom}(\tilde{G}, \mathbf{T})$ having kernel of order 3^n are those of the form $\pm 3^{n-4} \delta_1 + \gamma$, where γ is in the subgroup spanned by α and β . The action of a general automorphism of \tilde{G} on $\text{Hom}(\tilde{G}, \mathbf{T})$ is as follows, where m is either 0 or 1, i is either 1 or 2, and j is 0, 1, or 2.

$$\begin{aligned}\delta_1 &\mapsto (-1)^m \delta_1 - j\alpha \\ \beta &\mapsto (-1)^m \beta + j\alpha \\ \alpha &\mapsto i\alpha\end{aligned}$$

It follows that as orbit representatives among elements having kernel a 3-group we may take $3^{n-4} \delta_1$, $3^{n-4} \delta_1 + \beta$, $3^{n-4} \delta_1 - \beta$ and $\delta_1 + \beta + \alpha$. The first of these has kernel containing a $(C_3)^3$ subgroup, while the others correspond to $G(n, -1)$, $G(n, 1)$ and $G'(4)$ respectively.

Proof. The only portion of the statement that we actually need in the sequel is the last sentence, which can be checked by a simple calculation, so we omit the proof. •

Remark. Instead of using $G(n, e)$ to construct \tilde{G} , one could start from \tilde{G} and use Proposition 3 to define $G(n, e)$. The obligation to provide a proof for Proposition 3 would then be greater, and one would still have to write down presentations for the groups $G(n, e)$, so this approach would not save any labour.

Cohomology.

The cohomology of the Lie group \tilde{G} will be calculated below, using the spectral sequence for \tilde{G} as an extension with kernel \tilde{P} and quotient cyclic of order three. From this the cohomology of $G(n, e)$ can be calculated easily. To minimise the number of letters employed to represent cohomology classes we adopt the following convention.

Notation. If ξ represents an element of $H^*(\tilde{G})$, we shall use the same symbol to represent its image in $H^*(\tilde{P})$, $H^*(\tilde{M})$, or $H^*(\tilde{N})$. In case of ambiguity we shall refer to the element ξ of $H^*(\tilde{X})$ as $\xi(\tilde{X})$. If we define elements $\xi(\tilde{P})$ and $\xi(\tilde{M})$, we do not wish to imply that these elements are images of an element $\xi(\tilde{G})$, but merely that their images in $H^*(\tilde{N})$ are equal. This convention extends in the obvious way to other subgroups of \tilde{G} .

We now define generators for $H^*(\tilde{M})$, and use them to define elements of $H^*(\tilde{G})$ which will later be shown to form a generating set.

Proposition 4. Define an element τ (resp. β , γ) in $\text{Hom}(\tilde{M}, \mathbf{T}) \cong H^2(\tilde{M})$ by insisting that it maps $Y^j Z^k t$ to t (resp. ω^j , ω^k), and let μ be any non-zero element of $H^3(\tilde{M})$. Then $H^*(\tilde{M})$ is generated by τ , β , γ and μ , subject only to the relations $3\beta = 0$, $3\gamma = 0$, $3\mu = 0$ (and of course any relations implied by anticommutativity). The action of conjugation by X on $H^*(\tilde{M})$ sends τ to $\tau + \gamma$, sends γ to $\gamma + \beta$, and fixes β and μ . The restriction map from $H^*(\tilde{M})$ to $H^*(M(n, e))$ is surjective and has kernel the ideal generated by $3^{n-3} \tau - e\beta$.

Proof. The cohomology of \tilde{M} is easily shown to be as claimed (note that $\tilde{M} \cong \mathbf{T} \oplus C_3 \oplus C_3$), as is the action of conjugation by X . In the Gysin sequence for BM as a \mathbf{T} -bundle over $B\tilde{M}$ the map from $H^n(\tilde{M})$ to $H^{n+2}(\tilde{M})$ is multiplication by the image in $H^2(\tilde{M})$ of $3^{n-4} \delta_1 - e\beta$, which is $3^{n-3} \tau - e\beta$. The kernel of multiplication by this element is trivial, which implies that the restriction map from $H^*(\tilde{M})$ to $H^*(M)$ is surjective, and its kernel is clearly as claimed. •

Definition/Proposition 5. In addition to the elements α , β and δ_1 of $H^2(\tilde{G})$ defined in the statement of Proposition 3, define elements δ_2 , δ_3 , ζ and μ of $H^*(\tilde{G})$ as follows. Let ρ be a 3-dimensional irreducible representation of \tilde{G} whose restriction to \tilde{M} contains the 1-dimensional representation with first Chern class

τ , and define δ_3 to be the third Chern class of ρ . Define δ_2 and ζ as transfers from \widetilde{M} by the following equations.

$$\delta_2 = \text{Cor}_{\widetilde{M}}^{\widetilde{G}}(\tau^2) \quad \zeta = \text{Cor}_{\widetilde{M}}^{\widetilde{G}}(\tau^2(\beta + \gamma)) = \delta_2\beta(\widetilde{G}) + \text{Cor}_{\widetilde{M}}^{\widetilde{G}}(\tau^2\gamma)$$

The element $\mu(\widetilde{G})$ may be uniquely defined by requiring that its image in $H^3(\widetilde{M})$ is $\mu(\widetilde{M})$. The following equation relates $\delta_1(\widetilde{G})$ and $\tau(\widetilde{M})$.

$$\delta_1 = \text{Cor}_{\widetilde{M}}^{\widetilde{G}}(\tau) - \beta$$

Proof. The last two sentences of the above statement require a proof, the rest being definitions. The assertion concerning $\mu(\widetilde{G})$ is equivalent to the assertion that $H^3(\widetilde{G})$ has order three and maps injectively to $H^3(\widetilde{M})$. This can be shown easily by considering the spectral sequence for \widetilde{G} as an extension with kernel \mathbf{T} and quotient E , but will also follow from our study of the spectral sequence for \widetilde{G} as an extension with kernel \widetilde{P} and quotient C_3 , so we postpone the proof of this statement until the end of the proof of Lemma 8.

To verify the equation relating $\text{Cor}_{\widetilde{M}}^{\widetilde{G}}(\tau)$ to δ_1 , note that the transfer from $H^2(\widetilde{M})$ to $H^2(\widetilde{G})$ is equal to the following composite,

$$H^2(\widetilde{M}) \cong \text{Hom}(\widetilde{M}, \mathbf{T}) \xrightarrow{t^*} \text{Hom}(\widetilde{G}_{\text{ab}}, \mathbf{T}) \cong H^2(\widetilde{G}),$$

where t^* is the map induced by the classical transfer map from $\widetilde{G}_{\text{ab}}$ to \widetilde{M} . Using this we may describe $\text{Cor}(\tau)$ as an element of $\text{Hom}(\widetilde{G}, \mathbf{T})$ by the following equations.

$$\begin{aligned} \text{Cor}(\tau)(X^i Y^j Z^k t) &= \tau(Y^j Z^k t) \tau(X Y^j Z^k t X^{-1}) \tau(X^{-1} Y^j Z^k t X) \\ &= \tau(Y^j Z^k t) \tau(Y^j Z^{j+k} t \omega^k) \tau(Y^j Z^{2j+k} t \omega^{2k+j}) \\ &= t^3 \omega^j \\ &= (\delta_1 + \beta)(X^i Y^j Z^k t) \end{aligned}$$

•

Theorem 6. Define $\gamma \in \text{Hom}(\widetilde{P}, \mathbf{T})$ by the equation $\gamma(X^i Z^k t) = \omega^k$, and let $\alpha, \delta_1, \delta_2, \delta_3$ be the restrictions to \widetilde{P} of the elements of $H^*(\widetilde{G})$ having the same names. Then these five elements generate $H^*(\widetilde{P})$ subject only to the following relations.

$$\begin{aligned} 3\gamma &= 3\alpha = 0, & \alpha^3 \gamma &= \gamma^3 \alpha, \\ \alpha \delta_1 &= 0 = \gamma \delta_1, & \alpha \delta_2 &= 0, & \gamma \delta_2 &= -\gamma^3 + \alpha^2 \gamma, \\ \delta_1^2 &= 3\delta_2, & \delta_1 \delta_2 &= 9\delta_3, & \delta_2^2 &= 3\delta_3 \delta_1 + \gamma^4 - \alpha^2 \gamma^2. \end{aligned}$$

The action of conjugation by Y on $H^*(\widetilde{P})$ sends γ to $\gamma - \alpha$ and fixes the other generators.

The image of the map from $H^*(\widetilde{P})$ to $H^*(P(n, e))$ is the whole of $H^{\text{even}}(P(n, e))$, and the kernel is the ideal generated by $3^{n-4}\delta_1$. As a module for $H^{\text{even}}(P(n, e))$, $H^{\text{odd}}(P(n, e))$ is generated by two elements μ_1 and μ_2 of degree three, subject to the following relations.

$$\begin{aligned} 3\mu_1 &= 3\mu_2 = 0, & \mu_1 \delta_1 &= \mu_2 \delta_1 = 0, & \mu_1 \gamma &= \mu_2 \alpha, \\ \mu_1 \delta_2 &= 0, & \mu_2 \delta_2 &= -\gamma^2 \mu_2 - \alpha^2 \mu_2, & \alpha^3 \mu_2 &= \gamma^3 \mu_1. \end{aligned}$$

The ring structure of $H^*(P(n, e))$ is determined by the above relations together with the relation

$$\mu_1 \mu_2 = \begin{cases} 0 & \text{for } n > 4 \\ 3\delta_3 & \text{for } n = 4. \end{cases}$$

Proof. The integral cohomology of \widetilde{P} and $P(n, e)$ is computed in [9], where the group $P(n, e)$ is called $P(n-1)$, and a different generator is taken in degree four from the one used above. Assuming the statements contained in [9], we only need to compare the restrictions to \widetilde{P} of our generators for $H^*(\widetilde{G})$ and the generators taken for $H^*(\widetilde{P})$ in [9]. The representation used to define $\delta_3(\widetilde{G})$ restricts to \widetilde{P} as the representation used there to define the degree six generator in $H^*(\widetilde{P})$. The behaviour of the degree two generators defined as homomorphisms when restricted to \widetilde{P} is clear. For the generators defined in terms of the transfer the required relations follow from the observation that $\widetilde{M}\widetilde{P} = \widetilde{G}$, and so the double coset formula gives the equation $\text{Cor}_{\widetilde{N}}^{\widetilde{P}} \text{Res}_{\widetilde{N}}^{\widetilde{M}} = \text{Res}_{\widetilde{P}}^{\widetilde{G}} \text{Cor}_{\widetilde{M}}^{\widetilde{G}}$. •

Proposition 7. *The restrictions to \widetilde{M} and \widetilde{P} of the elements of $H^*(\widetilde{G})$ of Definition 5 are either of the form “element maps to element having the same name”, or are included in the following lists.*

Restrictions to \widetilde{P} :

$$\text{Res}(\beta) = 0, \quad \text{Res}(\mu) = 0, \quad \text{Res}(\zeta) = \alpha^2\gamma - \gamma^3.$$

Restrictions to \widetilde{M} :

$$\text{Res}(\alpha) = 0, \quad \text{Res}(\delta_1) = 3\tau, \quad \text{Res}(\zeta) = \beta^2\gamma - \gamma^3,$$

$$\text{Res}(\delta_2) = 3\tau^2 - \tau\beta - \gamma^2 + \gamma\beta + \beta^2, \quad \text{Res}(\delta_3) = \tau^3 + \tau^2\beta - \tau\gamma^2 + \tau\gamma\beta.$$

Proof. For the elements defined using the transfer, the double coset formula suffices to obtain the above results. As an example, the restriction to \widetilde{P} of ζ may be found as follows.

$$\begin{aligned} \text{Res}_{\widetilde{P}}^{\widetilde{G}}(\zeta) &= \text{Res}_{\widetilde{P}}^{\widetilde{G}} \text{Cor}_{\widetilde{M}}^{\widetilde{G}}(\tau^2(\beta + \gamma)) \\ &= \text{Cor}_{\widetilde{N}}^{\widetilde{P}} \text{Res}_{\widetilde{N}}^{\widetilde{M}}(\tau^2(\beta + \gamma)) \\ &= \delta_2(\widetilde{P})\gamma(\widetilde{P}) = \alpha^2\gamma - \gamma^3. \end{aligned}$$

The restriction to \widetilde{M} of the representation ρ used to define δ_3 contains a summand with Chern class τ , so must also contain summands with Chern classes $\tau + \gamma$ and $\tau - \gamma + \beta$, these being images of τ under the action of conjugation by powers of X on $H^*(\widetilde{M})$. Its third Chern class (which is by definition the restriction to \widetilde{M} of δ_3) is the product of these three elements. •

Lemma 8. *The spectral sequence with integer coefficients for \widetilde{G} as an extension with kernel \widetilde{P} and quotient of order three collapses. The seven elements of $H^*(\widetilde{G})$ of Definition 5 generate the E_2 -page. The elements $\alpha, \delta_1, \delta_2, \delta_3$ and ζ yield elements in $E_2^{0,*}$, β yields a generator for $E_2^{2,0}$, and μ yields a generator for $E_2^{1,2}$. The map of spectral sequences induced by the diagram*

$$\begin{array}{ccccc} \widetilde{N} & \longrightarrow & \widetilde{M} & \longrightarrow & C_3 \\ \downarrow & & \downarrow & & \downarrow \\ \widetilde{P} & \longrightarrow & \widetilde{G} & \longrightarrow & C_3 \end{array}$$

is injective on the $E_2^{i,j}$ such that $i + j$ is odd. The ring structure of the E_2 -page is given by the following relations.

$$\begin{aligned} 3\alpha &= 3\beta = 0, & 3\mu &= 0, & 3\zeta &= 0, \\ \alpha\delta_1 &= 0, & \delta_1^2 &= 3\delta_2, & \alpha\delta_2 &= 0, & \delta_1\delta_2 &= 9\delta_3, \\ \alpha\zeta &= 0, & \delta_1\zeta &= 0, & \alpha^2\beta &= 0, \\ \alpha\mu &= 0, & \delta_1\mu &= 0, & 27\delta_3^2 - \delta_2^3 &= \zeta^2. \end{aligned}$$

Proof. First we split $H^*(\widetilde{P})$ as a sum of indecomposable modules for the action of C_3 . From the relations given in Theorem 6 it is easy to show that the elements $\delta_3^i\delta_1, \delta_3^i\delta_2, \delta_3^i\gamma^j, \delta_3^i\gamma^j\alpha, \delta_3^i\gamma^j\alpha^2, \delta_3^i\alpha^{j+3}$ form a basis for $H^*(\widetilde{P})$, where i and j are any positive integers. (Recall that we have defined a basis for an abelian group to be a set of elements not containing the identity element such that the group is equal to the direct sum of the cyclic subgroups generated by those elements.) The C_3 -submodules spanned by $\delta_3^i, \delta_3^i\delta_1$ and $\delta_3^i\delta_2$ are direct summands isomorphic to the trivial C_3 -module \mathbf{Z} , and the monomials of fixed degree in δ_3 and fixed total degree in α and γ (there are at most four such for any choice of degrees) form a C_3 -summand. The elements $\delta_3^i\alpha$ and $\delta_3^i\gamma$ form an indecomposable C_3 -summand with underlying group $C_3 \oplus C_3$, and the elements $\delta_3^i\alpha^2, \delta_3^i\alpha\gamma, \delta_3^i\gamma^2$ form an indecomposable C_3 -summand which must be a free \mathbf{F}_3C_3 -module of rank one. For $j \geq 3$ and for any i , the \mathbf{F}_3C_3 -module generated by the four monomials of degree i in δ_3 and degree j in α and γ splits as a direct sum of a trivial module generated by $\delta_3^i(\gamma^j - \gamma^{j-2}\alpha^2)$ and an indecomposable module containing $\delta_3^i\alpha^j, \delta_3^i\gamma^{i-1}\alpha$ and $\delta_3^i\gamma^{j-2}\alpha^2$, which is therefore free.

Using the above C_3 -splitting of $H^*(\tilde{P})$, it is easy to check that the elements $\delta_3^i \delta_1$, $\delta_3^i \delta_2$, $\delta_3^i \alpha^j$ and $\delta_3^i (\gamma^{j+3} - \gamma^{j+1} \alpha^2)$ where $i, j \geq 0$ form a basis for the fixed point subring. Identifying elements of $H^*(\tilde{G})$ with their images in $H^*(\tilde{P})$ and applying Theorem 6 and Proposition 7, we see that for $i \geq 0$ the following equalities hold.

$$\begin{aligned}\gamma^{3i+3} - \gamma^{3i+1} \alpha^2 &= \zeta^{i+1} \\ \gamma^{3i+4} - \gamma^{3i+2} \alpha^2 &= \zeta^i (\delta_2^2 - 3\delta_3 \delta_1) \quad (= \zeta^i \delta_2^2 \text{ if } i > 0) \\ \gamma^{3i+5} - \gamma^{3i+3} \alpha^2 &= -\zeta^{i+1} \delta_2\end{aligned}$$

It is now easy to see that the elements $\delta_3^i \delta_1$, $\delta_3^i \delta_2$, $\delta_3^i \alpha^j$, $\delta_3^i \zeta^{j+1}$, $\delta_3^i \zeta^{j+1} \delta_2$, $\delta_3^i (\delta_2^2 - 3\delta_3 \delta_1)$ and $\delta_3^i \zeta^{j+1} \delta_2^2$ where $i, j \geq 0$ also form a basis for the fixed point subring. This already implies that all differentials in the spectral sequence are trivial on $E^{0,*}$. It is now easy to see that the relations claimed between the elements α , δ_1 , δ_2 , δ_3 and ζ are exactly the relations that do hold between them as elements of $H^*(\tilde{P})^{C_3}$, or equivalently as elements of $E_2^{0,*}$. Note that the fact that $\delta_2^2 - 3\delta_3 \delta_1$ has order three follows from the given relations by expanding $\delta_1^2 \delta_2$ in two different ways.

It is clear that in the spectral sequence β yields an element of $E_2^{2,0}$ because as a homomorphism from \tilde{G} to \mathbf{T} , β has kernel \tilde{P} . Cup product with this element of $E_2^{2,0}$ gives a surjection from $E_2^{0,j}$ to $E_2^{2,j}$ and an isomorphism from $E_2^{i+1,j}$ to $E_2^{i+3,j}$ for all i . We now consider the cohomology of the quotient C_3 with coefficients in the various modules that occur in our decomposition of $H^*(\tilde{P})$. Of the types of module that occur, each has second cohomology group of order 3 except for the free $\mathbf{F}_3 C_3$ -module, which has trivial positive degree cohomology. The subset of the above basis of $H^*(\tilde{P})^{C_3}$ corresponding to free $\mathbf{F}_3 C_3$ -summands of $H^*(\tilde{P})$ consists of the elements $\zeta^i \alpha^{j+2}$, so our relations between the elements of $E_2^{*,*}$ of even total degree may be completed by adding the relations $3\beta = 0$ and $\alpha^2 \beta = 0$.

Now we consider the elements of $E_2^{*,*}$ of odd total degree. The first cohomology group of C_3 with coefficients in \mathbf{Z} or in $\mathbf{F}_3 C_3$ is trivial, and the corresponding group with coefficients in each of the other two types of indecomposable summands of $H^*(\tilde{P})$ is cyclic of order three. Using the above C_3 -splitting of $H^*(\tilde{P})$ it is now easy to find the dimension over \mathbf{F}_3 of $E_2^{1,j}$. If we define $P(t)$ to be the power series whose coefficient of t^j is the dimension of $E_2^{1,j}$, then the following equation describes $P(t)$.

$$P(t) = (t^6 - t^4 + t^2)/(1 - t^6)(1 - t^2)$$

It is reasonably easy to describe the rest of the multiplicative structure of the E_2 -page (that is, the products involving at least one element of odd total degree) directly, but this will follow from the assertion concerning restriction to the spectral sequence for \tilde{M} expressed as an extension with kernel \tilde{N} and quotient of order three. Let $\overline{E}_*^{*,*}$ be the spectral sequence for this extension. Then $\overline{E}_*^{*,*}$ collapses, and the E_2 -page is isomorphic as a ring to $H^*(\tilde{M})$, where τ and γ yield elements of $\overline{E}_2^{0,2}$, β yields an element of $\overline{E}_2^{2,0}$, and μ yields an element of $\overline{E}_2^{1,2}$. It is easy to check that $E_2^{1,2}$ maps isomorphically to $\overline{E}_2^{1,2}$ using the cohomology long exact sequence associated to the following short exact sequence of C_3 -modules, where K (which is defined by this sequence) is a trivial module isomorphic to the integers modulo three.

$$0 \longrightarrow K \longrightarrow H^2(\tilde{P}) \longrightarrow H^2(\tilde{N}) \longrightarrow 0$$

The images of α , δ_1 , δ_2 , δ_3 and ζ in $\overline{E}_2^{0,*}$ are just the coefficients of β^0 in their restrictions to \tilde{M} (see Proposition 7). In $\overline{E}_2^{*,*}$, $\alpha = 0$, and δ_1 is divisible by three, so here the relations $\alpha\mu = 0$ and $\delta_1\mu = 0$ hold. The subring of $E_2^{0,*}$ generated by δ_2 , δ_3 and ζ maps injectively to $\overline{E}_2^{0,*}$ (as can be seen by checking the images of the basis for this ring given earlier), and we shall temporarily refer to this subring of $\overline{E}_2^{0,*}$ as R . The R -module generated by μ is isomorphic to $R/3R$, and it may be checked that the Poincaré series for this subgroup of $\overline{E}_2^{1,*}$ is equal to the series $P(t)$ above. From this it follows that the odd total degree subgroups of $E_2^{*,*}$ map injectively to $\overline{E}_2^{*,*}$, that no other generators are required for E_2 , and that the relations in E_2 involving μ are as claimed.

To show that the spectral sequence $E_2^{*,*}$ collapses we must show that each of the seven generators survives. From Definition 5 it is clear that each of the even degree generators actually comes from an

element of $H^*(\tilde{G})$, so must survive. The definition given there for μ is more nebulous, and it has not yet been shown that such an element exists, although the preceding paragraph shows that such an element is unique. The only way for $\mu \in E_2^{1,2}$ to fail to survive is for $d_3(\mu)$ to be a non-zero multiple of β^2 , and there are many ways to see that this cannot happen. For example, in $\overline{E}_*^{*,*}$, $d_3(\mu)$ is zero, but $E_3^{4,0}$ maps injectively to $\overline{E}_3^{4,0}$, so $d_3(\mu)$ must be zero in $E_*^{*,*}$ too. Alternatively one may use the fact that for any split extension and any trivial coefficients, no differential in the spectral sequence can hit the base-line (see [12], where the condition that the coefficients be trivial is omitted). This completes the proofs of Lemma 8 and Definition 5.

•

Corollary 9. *Chern classes of representations generate the even degree cohomology of $G(n, e)$ and $G'(4)$.*

Proof. By choosing a slightly different set of generators for $H^*(\tilde{G})$ from the one we have chosen, and using the arguments of Lemma 8, it is easy to show that Chern classes generate the even degree cohomology of \tilde{G} . It is also known that Chern classes generate the even degree cohomology of the subgroups $M(n, e)$ and the abelian maximal subgroup of $G'(4)$. Applying one implication of Lemma 1 to \tilde{M} , it may be seen that multiplication by the Chern class of the **T**-bundle $BM(n, e)$ over $B\tilde{M}$ is injective on the odd degree cohomology of \tilde{M} . However, it was shown in Lemma 8 that the odd degree cohomology of \tilde{G} maps injectively to that of \tilde{M} , and it follows that multiplication by the Chern class of the **T**-bundle $BG(n, e)$ (resp. $BG'(4)$) over $B\tilde{G}$ is injective on the odd degree cohomology of \tilde{G} . Now the other implication of Lemma 1 applied to \tilde{G} gives the required result. •

Remark. In the introduction we remarked that there are two groups of order p^n for each prime $p \geq 5$ and each $n \geq 4$ having similar presentations to the groups $G(n, e)$. These groups occur as normal subgroups of the (unique) Lie group having p^3 circular components and nilpotence class three, just as $G(n, e)$ occurs within \tilde{G} . This Lie group is expressible as a split extension with kernel the unique non-abelian Lie group having p^2 circular components, and quotient of order p . I do not know whether the spectral sequence for this extension collapses, but the methods of Lemma 8 can be used to show that for this Lie group corestrictions of Chern classes generate the even degree cohomology, and that the odd degree cohomology maps injectively to the cohomology of its (unique) abelian maximal subgroup. As in Corollary 9 it is possible to deduce that for the corresponding finite groups, corestrictions of Chern classes generate the even degree cohomology. This result has been obtained using other methods by N. Yagita [19]. It may also be shown that for these groups Chern classes alone do not suffice to generate the even degree cohomology [11].

Theorem 10. *The integral cohomology of the Lie group \tilde{G} is generated by the seven elements of Definition 5 subject to the following relations.*

$$\begin{aligned} 3\alpha = 3\beta = 0, \quad 3\mu = 0, \quad 3\zeta = 0, \\ \alpha\delta_1 = -\alpha\beta, \quad \delta_1^2 = 3\delta_2\beta, \quad \alpha\delta_2 = 0, \quad \delta_1\delta_2 = 9\delta_3, \\ \alpha\zeta = 0, \quad \delta_1\zeta = 0, \quad \alpha^2\beta = -\delta_1\beta^2, \\ \alpha\mu = 0, \quad \delta_1\mu = 0, \\ \delta_2^3 - 27\delta_3^2 + \zeta^2 = -\delta_3(\delta_1\beta^2 + \beta^3) + \delta_2^2\beta^2 + \delta_2\beta^4 - (\delta_1\beta^5 + \beta^6). \end{aligned}$$

Proof. Filtering the ring given by the above relations by powers of β one obtains the ring of Lemma 8, and so if these relations hold, then they suffice. It remains to show that these relations do hold. The relations $(\delta_1 + \beta)\alpha = 0$, $\alpha\zeta = 0$ and $\alpha\delta_2 = 0$ follow easily using Frobenius reciprocity, because $\delta_1 + \beta$, ζ and δ_2 are defineable as corestrictions from \tilde{M} while α restricts to \tilde{M} as zero. The relations involving μ follow easily from the fact that the odd degree cohomology of \tilde{G} maps injectively to that of \tilde{M} . The expressions for δ_1^2 , $\delta_1\delta_2$ and $\zeta\delta_1$ may also be shown to hold using Frobenius reciprocity. As an example, the following equations verify the expression for δ_1^2 .

$$\begin{aligned} \delta_1^2 &= (\text{Cor}_{\tilde{M}}^{\tilde{G}}(\tau) - \beta)\delta_1 = \text{Cor}_{\tilde{M}}^{\tilde{G}}(\tau)\delta_1 - \delta_1\beta \\ &= \text{Cor}(3\tau^2) - \delta_1\beta = 3\delta_2 - \delta_1\beta \end{aligned}$$

From Lemma 8 we see that $\alpha^2\beta = a\delta_1\beta^2 + b\alpha\beta^2 + c\beta^3$ for some a, b and c . By considering the image of this equation in \tilde{M} it may be shown that $c = 0$. As homomorphisms from \tilde{G} to \mathbf{T} , α and β have kernel containing \mathbf{T} , so they are in the image of the inflation from $E = \tilde{G}/\mathbf{T}$. It follows from Theorem 6 that $\alpha^3\beta = \beta^3\alpha$. It is now possible to solve for a and b using the following equations to show that either $\alpha^2\beta = -\delta_1\beta^2$ or $\alpha^2\beta = \pm\alpha\beta^2$.

$$\beta^3\alpha = \alpha^3\beta = \alpha(a\delta_1\beta^2 + b\alpha\beta^2) = -a\alpha\beta^3 + b\alpha^2\beta^2 = (b^2 - a)\alpha\beta^3 + ab\delta_1\beta^3$$

It remains to rule out the possibility that $\alpha^2\beta = \pm\alpha\beta^2$. The kernel of $\alpha + \beta$ viewed as a map from \tilde{G} to \mathbf{T} is a group isomorphic to \tilde{P} , and $\alpha, \beta, \alpha - \beta$ each restrict to this subgroup as a non-zero element of order three. Theorem 6 shows that the subring of $H^*(\tilde{P})$ generated by elements of H^2 of order three is isomorphic to $\mathbf{Z}[x, y]/(3x, 3y, x^3y - y^3x)$. In this ring the product of any three non-zero elements of degree two (the degree of x and y is non-zero). It follows that $\alpha^2\beta - \alpha\beta^2$ is non-zero in $H^*(\tilde{G})$ because its restriction to $\ker(\alpha + \beta)$ is non-zero. Similarly, $\alpha^2\beta + \alpha\beta^2$ is non-zero because its restriction to $\ker(\alpha - \beta)$ is non-zero.

There remains now only the expression given for $\delta_2^3 - 27\delta_3^2 + \zeta^2$. Lemma 8 implies that this quantity is a multiple of β , and the relations we have already obtained show that it is annihilated by α . Using Lemma 8, a basis for βH^{10} may be found (this group is isomorphic to $(C_3)^{11}$). Using the relations already known it may be shown that the kernel of multiplication by α on βH^{10} has basis $\delta_3\delta_2\beta, \zeta\delta_2\beta, \delta_2^2\beta^2, \delta_3(\delta_1\beta^2 + \beta^3), \zeta\beta^3, \delta_2\beta^4, \delta_1\beta^5 + \beta^6$, and using Proposition 7 it may be shown that this group maps injectively to $H^{12}(\tilde{M})$. The relation claimed now holds, because when multiplied by α it gives the valid relation $0 = 0$, and its image in $H^{12}(\tilde{M})$ is also a valid relation. •

Corollary 11. *The groups of order 81 are distinguished by their integral cohomology groups.*

Proof. The hardest part of the proof is to distinguish $G(4, 1)$, $G(4, -1)$ and $G'(4)$, so we shall only sketch the other cases. There are 15 groups of order 3^4 (see [3] for a classification, or [17] for presentations). First we use $H^2(G) \cong \text{Hom}(G, \mathbf{T})$ to distinguish the five abelian groups, and to split the non-abelian groups into three classes of sizes three, three and four, with respective H^2 groups $C_3 \oplus C_9$, $(C_3)^3$ and $(C_3)^2$. The groups G such that $H^2(G) \cong C_3 \oplus C_9$ are two split metacyclic groups, only one of which contains an element of order 27, and one non-metacyclic. Three easy spectral sequence arguments show that the metacyclic with an element of order 27 has $H^3 = 0$, the other metacyclic has $H^3 \cong C_3$, and the third group has $H^3 \cong C_3 \oplus C_3$. The groups with $H^2 \cong (C_3)^3$ are the direct products of C_3 with each of the two non-abelian groups of order 27 and another group isomorphic to the subgroup $P(5, e)$ of $G(5, e)$. Applying the Künneth theorem and Lewis' description of the cohomology rings of the groups of order p^3 [12], we see that $H^3 \cong (C_3)^4$ for the product of C_3 and the group of order 27 and exponent three, while $H^3 \cong (C_3)^2$ for the other two groups. These two groups may be distinguished using H^4 , because for $P(5, e)$ this group has exponent nine, while for the product it has exponent three.

There remain only the four groups with $H^2 \cong (C_3)^2$. These occur as normal subgroups of \tilde{G} with connected quotient, so we use the Gysin sequence for $B\tilde{G}$ as a \mathbf{T} -bundle over $B\tilde{G}$ to study $H^*(G)$. Using Theorem 10 we may write down a basis for the first few cohomology groups of \tilde{G} as follows.

$$\begin{array}{ll} H^2 & \delta_1, \alpha, \beta \\ H^3 & \mu \\ H^4 & \delta_2, \alpha^2, \delta_1\beta, \alpha\beta, \beta^2 \\ H^5 & \beta\mu \\ H^6 & \delta_3, \alpha^3, \zeta, \delta_2\beta, \delta_1\beta^2, \alpha\beta^2, \beta^3 \end{array}$$

Let ξ be one of the elements $\delta_1 - \beta, \delta_1 + \beta, \delta_1 + \beta + \alpha$, or δ_1 . (Recall from Proposition 3 that the kernels of these four elements viewed as maps from \tilde{G} to \mathbf{T} are in distinct $\text{Aut}(\tilde{G})$ classes of normal subgroup, and that they are isomorphic to $G(4, 1)$, $G(4, -1)$, $G'(4)$ and the wreath product of C_3 with C_3 respectively.) In each case multiplication by ξ is injective from H^2 to H^4 , except the case $\xi = \delta_1 + \beta$, when the kernel has order three. Thus each of the four groups has H^3 of order three, except $G(4, -1)$ which has H^3 of order nine. In the remaining three cases, $H^4/\xi H^2$ has order 27. For each of these except the case $\xi = \delta_1$ multiplication by ξ is an isomorphism from H^3 to H^5 . Thus the wreath product of C_3 with C_3 has H^4 of order 81, while

$G(4, 1)$ and $G'(4)$ have H^4 of order 27. Now consider the remaining two cases. Multiplication by $\delta_1 + \beta + \alpha$ is injective from H^4 to H^6 , whereas multiplication by $\delta_1 - \beta$ has kernel of order three (generated by $\delta_1\beta + \alpha^2$). Hence $H^5(G(4, 1)) \cong C_3$ whereas $H^5(G'(4)) = 0$. •

Remarks. Lluís and Cárdenas have found the additive structure of the cohomology of C_3 wreath C_3 [4], and many authors have studied the cohomology of the metacyclic groups. A similar result to Corollary 11 holds for the groups of order 16, but N. Yagita has exhibited a pair of groups of order p^4 for all $p \geq 5$ having isomorphic integral cohomology groups [19]. It is not known whether these groups have isomorphic cohomology rings.

It will be seen shortly that for $n \geq 5$ the integral cohomology of $G(n, e)$ is generated by the image of $H^*(\tilde{G})$ and one other element in degree 5. The following proposition describes such an element.

Lemma 12. *There is an element ν of $H^5(G(n, e))$ which restricts to $M(n, e)$ as $\gamma\mu$. The element ν is not in the image of $H^5(\tilde{G})$, and may be chosen to be a corestriction from $H^5(P(n, e))$.*

Proof. Proposition 7 describes the map from $H^*(\tilde{G})$ to $H^*(\tilde{M})$, and Proposition 4 describes the map from $H^*(\tilde{M})$ to $H^*(M)$. Together these show that the image of $H^5(\tilde{G})$ in $H^5(M)$ is trivial, and hence that if ν exists it cannot be in the image of $H^5(\tilde{G})$. To show that ν exists, we use the double coset formula $\text{Res}_M^G \text{Cor}_P^G = \text{Cor}_N^M \text{Res}_N^P$. It is easy to check that $H^*(N)$ is generated by τ , γ and μ' of degrees 2, 2 and 3 respectively, where τ and γ are the images of the elements of $H^*(M)$ of the same name, while μ' satisfies the equation $\text{Cor}_N^M(\mu') = \mu$ (note that $\text{Res}_N^M(\mu) = 0$). It follows by Frobenius reciprocity that $\text{Cor}_N^M(\gamma\mu') = \gamma\mu$, and so it remains to prove that $\gamma\mu'$ is in the image of the restriction from P to N . The element γ is the image of the element of $H^2(P)$ having the same name. To show that $H^3(P)$ maps onto $H^3(N)$, consider the map of spectral sequences induced by the following commutative diagram.

$$\begin{array}{ccccc} \mathbf{T} & \longrightarrow & BN & \longrightarrow & B\tilde{N} \\ \downarrow \text{Id} & & \downarrow & & \downarrow \\ \mathbf{T} & \longrightarrow & BP & \longrightarrow & B\tilde{P} \end{array}$$

For each spectral sequence, $E_3^{3,0} = 0$, and the induced map is surjective on $E_3^{2,1}$, which implies that $H^3(P)$ maps onto $H^3(N)$. •

Theorem 13. *For $n \geq 5$, the integral cohomology ring of $G(n, e)$ is generated by elements α , δ_1 , μ , δ_2 , ν , δ_3 , ζ of degrees 2, 2, 3, 4, 5, 6 and 6 respectively. The element ν is as described in Lemma 12, and the other generators are the restrictions of the elements of $H^*(\tilde{G})$ (see Theorem 10) having the same names. They are subject only to the following relations.*

$$\begin{aligned} 3\alpha &= 0, & 3\mu &= 0, & 3\nu &= 0, & 3\zeta &= 0, \\ 3^{n-3}\delta_1 &= 0, & 3^{n-2}\delta_2 &= 0, & 3^{n-1}\delta_3 &= 0, \\ 3\delta_2 &= \delta_1^2(1 + e3^{n-4}), & \delta_1\alpha &= 0, & \alpha\mu &= 0, \\ 9\delta_3 &= \delta_1\delta_2, & \delta_1\mu &= 0, & \delta_2\alpha &= 0, \\ \zeta\delta_1 &= 0, & \zeta\alpha &= 0, & \delta_2^3 &= 27\delta_3^2 - \zeta^2, \\ \delta_1\nu &= 0, & \delta_2\nu &= \zeta\mu, & \zeta\nu &= -\delta_2^2\mu, & \mu\nu &= 0. \end{aligned}$$

Proof. From the Gysin sequence for $BG(n, e)$ as a \mathbf{T} -bundle over $B\tilde{G}$, whose differential is described in Proposition 3, it follows that $H^m(G(n, e))$ is expressible as an extension with kernel $H^m(\tilde{G})/\xi H^{m-2}(\tilde{G})$ and quotient $\ker(\times\xi : H^{m-1}(\tilde{G}) \rightarrow H^{m+1}(\tilde{G}))$, where $\xi = 3^{n-4}\delta_1 - e\beta$. To find the kernel of multiplication by ξ , it is helpful to note that the only torsion of order larger than three in $H^*(\tilde{G})$ is generated by $\delta_3^i(\delta_2^2 - 3\delta_3\delta_1)$, which is annihilated by δ_1 , and so multiplication by $3^{n-4}\delta_1$ is trivial on the torsion subgroup of $H^*(\tilde{G})$. Moreover, it is easy to see that the kernel of multiplication by $3^{n-4}\delta_1$ is exactly the torsion subgroup of $H^*(\tilde{G})$, and hence that $\ker(\times\xi)$ is the intersection of $\ker(\times\beta)$ and the torsion subgroup of $H^*(\tilde{G})$. It may now be checked that $\ker(\times\xi)$ is equal to the ideal of $H^*(\tilde{G})$ generated by $\alpha^2 + \delta_1\beta$ (here it helps to first find

the intersection of $\ker(\times\beta)$ and the torsion subgroup not in $H^*(\tilde{G})$, but in the ring given by the filtration of Lemma 8). It now follows that the element ν of Lemma 12 together with the generators for $H^*(\tilde{G})$ forms a generating set for $H^*(G)$.

Since ν was defined as the transfer of an element of order three, there are no additive extension problems to worry about. The relations given in the statement between the generators coming from $H^*(\tilde{G})$ are just those that hold in $H^*(\tilde{G})$, after substituting $e3^{n-4}\delta_1$ for β throughout. As a module for the subalgebra of $H^*(G)$ generated by δ_3 and α , the ideal generated by $\alpha^2 + \delta_1\beta$ is isomorphic to the polynomial module $\mathbf{F}_3[\delta_3, \alpha]$, and the product of $\alpha^2 + \delta_1\beta$ with each of δ_1 , μ , δ_2 and ζ is zero. It follows that the only new relations we need to introduce are expressions for $\delta_1\nu$, $\mu\nu$, $\delta_2\nu$ and $\zeta\nu$ not involving ν . Since the restriction from \tilde{G} to \tilde{M} is injective in odd degrees, it follows that the restriction from G to M is injective on the image of $H^{\text{odd}}(\tilde{G})$ in $H^{\text{odd}}(G)$, and so the expressions given for $\delta_1\nu$, $\delta_2\nu$ and $\zeta\nu$ may be verified by checking their images in $H^*(M)$ (as described by Proposition 7 and Lemma 12). The relation $\mu\nu$ follows using Frobenius reciprocity, because ν is a transfer from $P(n, e)$, and the product of any element of $H^5(P)$ and any element of $H^3(P)$ is zero. •

Corollary 14. *For $n \geq 5$, the groups $G(n, 1)$ and $G(n, -1)$ have isomorphic integral cohomology rings.*

Proof. The primed elements of $H^*(G(n, 1))$ defined by the equations

$$\begin{aligned}\alpha' &= \alpha, & \delta_1' &= \delta_1, & \mu' &= \mu, & \delta_2' &= (1 + 3^{n-4})\delta_2, \\ \nu' &= \nu & \zeta' &= \zeta, & \delta_3' &= (1 + 3^{n-4})\delta_3,\end{aligned}$$

generate $H^*(G(n, 1))$, and satisfy the same relations as the original generating set for $H^*(G(n, -1))$. To check that $\delta_2'^3 = 27\delta_3'^2 - \zeta'^2$, it helps to note that the equations

$$3\delta_2'^2 = \delta_2'\delta_1'^2(1 + e3^{n-4}) = 9\delta_3'\delta_1'$$

imply that $\delta_2'^2$ has order 3^{n-4} . •

Remarks. Of course, Corollary 14 leaves many questions unanswered, mainly of the form “Are $\mathcal{F}(G(5, 1))$ and $\mathcal{F}(G(5, -1))$ isomorphic?” for various functors \mathcal{F} . We list some of these questions below.

Other coefficient rings: It follows from Corollary 14 and the Künneth theorem that the cohomology groups of $G(5, 1)$ and $G(5, -1)$ are isomorphic for any trivial coefficients. What can be said about the ring structure of the cohomology of these two groups with coefficients $\mathbf{Z}/(3^m)$? The strongest result in this direction would be to show that $G(5, 1)$ and $G(5, -1)$ have isomorphic cohomology spectra. (The cohomology spectrum, which was introduced by Bockstein [2] and also studied by Palermo [13] consists of the integral and mod- m cohomology rings for all m , together with the projection maps and Bockstein maps between these rings.) If the cohomology spectra of $G(5, 1)$ and $G(5, -1)$ are not isomorphic, this might lead to examples of pairs of groups distinguishable by their integral cohomology rings, but not by their integral cohomology groups, using direct products of copies of $G(5, e)$.

Cohomology operations: One may ask if the isomorphism between the cohomology rings of $G(5, 1)$ and $G(5, -1)$ commutes with the action of the algebra of integral cohomology operations. The author does not know the answer to this question, but can show that there is an isomorphism between the two rings commuting with the action of the operation “projection to mod-3 coefficients, followed by the first Steenrod reduced power, followed by the Bockstein back to integer coefficients”. This operation is the first possibly non-zero differential in the Atiyah-Hirzebruch spectral sequence for a 3-group. One may also ask if Massey products are capable of distinguishing the cohomology rings of $G(5, 1)$ and $G(5, -1)$. Again I do not know the answer.

K -theory: The ring $K^0(BG(n, e))$ is described by Atiyah’s theorem [1] together with the presentation for the representation ring of $G(n, e)$ given in Proposition 2. I have been unable to decide whether or not the representation rings of $G(5, 1)$ and $G(5, -1)$ are isomorphic however. For any prime p , the two non-abelian groups of order p^3 have isomorphic representation rings, and hence isomorphic K -theory rings, although the Λ -ring structures on their representation rings are quite different. This is reflected in the fact that their cohomology rings are quite different [12]. Even including the Λ -ring structure, the representation rings of $G(5, 1)$ and $G(5, -1)$ look very similar.

Other primes: As remarked earlier, there is for each odd prime a family of pairs of groups similar to the groups $G(n, e)$, which may provide examples for other primes of p -groups having isomorphic integral cohomology rings. Yagita has been able to show (using these groups as examples) that there exist non-isomorphic groups of order p^4 for $p \geq 5$ having isomorphic integral cohomology groups [19]. Yagita expects to be able to resolve the question of whether these groups have isomorphic cohomology rings, which may even appear in the final version of [19]. In contrast, work of Rusin [15] implies that even among the groups of order 32 there are no two having isomorphic integral cohomology.

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